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Minimum calculation of an integral with differential constraints

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Abstract

The aim of this paper is to minimize a constrained integral in the form of differential equations with constraints.For this purpose, we have learned many methods of obtaining functions' extremum in $\mathbb{R}^n$ and functions' extremal in calculus of variations. Then, we consider optimal control theory and look at it as a classical development of Euler–Lagrange theory. In such development, we are dealing with acceptable functions that are less well-behaved compared to normal functions. All of these cases reach us to Pontryagin's maximum principle (P.M.P).

Key words: Minimizing, Integral, differential constraints

1. Introduction

1-1. Minimum of single variable function

Theorem: Assume that function of $f(x)$ is defined at interval of I from real number. Find points of I that in these points, function of $f(x)$ has minimum value of its own.

Necessary and sufficient condition for $a \in I$ being minimum point of $f(x)$, is that for any value of $a \in I$, $f(a) \le f(x)$ is being established for only $x = a$.

Example: Find function relative minimum and relative maximum of $f(x) = x^3 + 7x^2 - 5x$ using first derivative test.

Solution:

$\frac{1}{3}$ and (-5) are critical points of f function. Therefore, we will have following results in according to previous theorem.

Relative maximum: $f(-5) = 75$

Relative minimum: $f\left(\frac{1}{3}\right) = -\frac{23}{27}$

1-2. Minimum of multivariable function

Theorem: assume that function of $f(x_1, x_2, \ldots, x_n)$ is defined at region of K from $\mathbb{R}^n$. Find a point at region of K, in which function of $f(x) = f(x_1, x_2, \ldots, x_n)$ finds minimum value of its own.

Example: By assuming the function of $f(x,y)=x^2+y^4+1$, find minimum point of f, if available.

Solution:

$$\begin{cases} \dfrac{\partial f}{\partial x}=2x=0 \\ \dfrac{\partial f}{\partial y}=4y^3=0 \end{cases}$$

The only solution for this system of equations equals (0,0). Therefore, $f(0,0)=1$ is only possible minimum value of f function. On the other hand,

$$f(x,y)=x^2+y^4+1 \geq f(0,0)=1$$

Subsequently, function of f has minimum value of $f(0,0)=1$ at the point of (0,0).

Note:

If function of $f(x_1,x_2,...,x_n)$ is defined at the region of K from $\mathbb{R}^n$, Taylor series of this function, at the point of $a=(a_1,...,a_n)$, will be as follows

$$f(a+\varepsilon h)=f(a)+\varepsilon h^T \, gradf(a)+\frac{\varepsilon^2}{2}h^T H(a)h+O(\varepsilon^3) \qquad (1)$$

Where:

$$gradf(a)=\begin{bmatrix} \dfrac{\partial f}{\partial x_1}(a) \\ \dfrac{\partial f}{\partial x_2}(a) \\ . \\ . \\ . \\ \dfrac{\partial f}{\partial x_n}(a) \end{bmatrix} \qquad H(a)=\begin{bmatrix} \dfrac{\partial^2 f(a)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 f(a)}{\partial x_1 \partial x_2} & . \ . & \dfrac{\partial^2 f(a)}{\partial x_1 \partial x_n} \\ & . & & \\ & . & & \\ & . & & \\ & . & & \\ \dfrac{\partial^2 f(a)}{\partial x_n \partial x_1} & \dfrac{\partial^2 f(a)}{\partial x_n \partial x_2} & . \ . & \dfrac{\partial^2 f(a)}{\partial x_n \partial x_n} \end{bmatrix}$$

We name this Matrix of $H(a)$ as Hessian matrix of $f(a)$ function.

Theorem: we assume that $f(x)$ has defined at open region of K from R^n and is well-defined enough, so that equation (1) is established. Therefore, only condition for $f(x)$ being local minimum at the point of $a \in K$, is that for every h:

$$gradf=0 \qquad , \qquad h^T Hh>0$$

Where, gradient and Hessian matrices has been calculated.

2. Main Text

In general, minimizing problem of one integral with differential constraints, is as follows:

Equation of state: $\dot{x_1} = f_i(x, u)$ $\quad i = 1,2, \dots, n$

First state: $x(t_0) = x^0$ $\quad$ in definite time of t_0

Final state (goal): $x(t_1) = x^1$ $\quad$ in time of t_1

A cost equation: $J = \int_{t_0}^{t_1} f_0(x, u)dt$

And a series of acceptable controls of $u(t)$ (means that at the bounded region of u, gradually continues and limited) calculation of followings are desirable:

Calculations of optimal control of $u^*(t)$ and its correspondent path of $x^*(t)$ that guide system from x^0 to x^1, so that, in this order J will be minimized.

2-1. Pontryagin's maximum principle

Without violating the generalities of problem, we consider that n equals 2.

Therefore, wewill have following control system:

$\dot{x}_1 = f_1(x_1, x_2, u)$ $\qquad$ $\dot{x}_2 = f_2(x_1, x_2, u)$

We want to guide the system from (x_1^0, x_2^0), at the time of $t = t_0$, to (x_1^1, x_2^1), to the unknown time of t_1, that is gradually continues and limited, using acceptable control functions of $u(t)$ (a single-variable control function belonging to controls' series of U which is a closed interval on real number axis)

So that, $J = \int_{t_0}^{t_1} f_0(x_1, x_2, u)dt$ is minimized, also.

Classical theory proposition is to investigate the behavior of the numerical function of Hamiltonian,

$$H = \psi_0 f_0(x_1, x_2, u) + \psi_1 f_1(x_1, x_2, u) + \psi_2 f_2(x_1, x_2, u)$$

Where, values of ψ_i are applied for thefollowing equations:

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i} \quad , \quad i = 0,1,2 \qquad (1)$$

Where, x_0 is the solution for differential equation of $\dot{x}_0 = f_0(x_1, x_2, u)$ and satisfy first condition of $x_0(t_0) = 0$.

For this case, consider the following theorem:

Theorem 2.1.Assume that $u^*(t)$ is an acceptable control function and its correspondent path is $x^* = (x_1^*, x_2^*)$ which guide system from x^0, at the time of $t = t^0$, to x^1, at the unknown time of t_1.

For the purpose of u^* and x^* being optimal (which means that they minimize J) it is necessary for Non-zero vector of $\Psi = (\Psi_0, \Psi_1, \Psi_2)^T$ to be applied in the equation 1 and, also it is necessary for following numerical function of:

$$H(\Psi, x, u) = \Psi_0 f_0(x, u) + \Psi_1 f_1(x, u) + \Psi_2 f_2(x, u)$$

to be available. So that:

a) For every H; $t_0 \le t \le t_1$, it reaches to its maximum value toward u when $u = u^*(t)$.

b) $H(\psi^*, x^*, u^*) = 0$ and when $u = u^*(t)$, equation (1) solution is $\psi^*(t)$ at time of $t = t_1$ and $\Psi_0 \le 0$

Also, it can be shown that,

$$H(\psi^*(t), x^*(t), u^*(t)) = \text{Constant}$$

And,

$$\psi_0(t) = \text{Constant}$$

Consequently, for every point on optimal path, we have: $H = 0$ and $\psi_0(t) \le 0$.

Example: Investigate following simple one-dimensional problem of,

$$\dot{x}_1 = -x_1 + u$$

From $x_1 = a$, at the time of $t = 0$ to $x_1 = b$, at unknown time of t_1, so that $y = \frac{1}{2}\int_0^{t_1} u^2 dt$ is minimized.

Solution:

We have,

$x(t_0) = x(0) = a = x_0$
$x(t_1) = b = x_1$

$f_0(x_1, x_2, u) = \frac{1}{2}u^2$, $f_1(x_1, x_2, u) = -x_1 + u$, $f_2(x_1, x_2, u) = 0$

For a moment, we assume that no constraints are applied on values that $u$can possess.
We write Hamilton function for variables of ψ_0, ψ_1 in same state,

$$H = \psi_0 \frac{u^2}{2} + \psi_1(-x_1 + u)$$

where,

$$\dot{\psi}_0 = -\frac{\partial H}{\partial x_0} \quad , \quad \dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = \psi_1$$

$\psi_0 = \text{Constant}$

This case will always be the same, because H will never be function of cost variable of x_0.

Therefore, $c\omega$ will only have one solution for every non-zero constant of c. Provided that requirement of theorem (1.2) is fulfilled, which is being non-negative value for ψ_0, therefore, we can choose any value we want for ψ_0. Therefore, $\psi_0 = -1$ will be chosen in this problem and any other application of theorem (1.2).
Now, we can solve equations in the same state.

$\psi_0 = -1$

A is a constant number.
By looking to its maximization as a function of u, we are now testing H.
There are no constraints for u, therefore, the derivate of H with respect to u is as follows:

$$H = -\frac{u^2}{2} + Ae^t(-x_1 + u)$$

$$\frac{\partial H}{\partial u} = -u + Ae^t = 0 \Rightarrow u = Ae^t$$

Therefore, for $u = \psi_1 = Ae^t$, H becomes extremum as a function of u.

Now, because of $\frac{\partial^2 H}{\partial u^2} = -1 < 0$

for$u = \psi_1 = Ae^t$, Hwill become extremum.

Therefore, we have $u(t) = \psi_1 = Ae^t$. Now, we will find correspondent path of u, that is x.

In other words, we find correspondentsolutionfor x_1.

$$\dot{x}_1 = -x_1 + Ae^t \Rightarrow \dot{x}_1 + x_1 = Ae^t$$

After solving, we would have following first-order linear differential equations.

$$x_1 = Be^{-t} + \frac{1}{2}Ae^t$$

Now, we would apply boundary conditions (in order to specify A and B in better path)

$$\begin{cases} x_1 = a \ , \ t = 0 \Rightarrow \quad a = B + \dfrac{A}{2} \\ x_1 = a \ , \ t_1 \quad \Rightarrow \quad b = Be^{-t_1} + \dfrac{A}{2}e^{t_1} \end{cases} \qquad (1)$$

In order to calculate A and B from above system of equations, we must obtain t_1 ,first.

For this purpose, t_1, time of reaching to target, is obtained on the optimal path using condition of $H = (\psi^*, x^*, u^*) = 0$.

Because of H being zero in all points on the optimal path, therefore we can apply it on everywhere we want that is appropriate. In this case, it is better to apply it at the time of t on optimal path. Therefore by substituting general solution of $x_1, \psi_1, u \ for \ H = 0$, we would have:

$$-\frac{1}{2}A^2e^{2t} + Ae^t(-Be^{-t} - \frac{A}{2}e^t + Ae^t) = 0$$

$$-\frac{1}{2}A^2e^{2t} - ABe^0 - \frac{A^2e^{2t}}{2} + A^2e^{2t} = 0$$

$$\Rightarrow AB = 0$$

Now, we test two special states;

A) $b = 2 \ , \ a = 1$

$$AB = 0 \begin{cases} A = 0 \xrightarrow{(a=1, b=2), (1), B=1} e^{-t_1} = 2 \\ or \\ B = 0 \xrightarrow{(a=1, b=2), (1), A=2} e^{t_1} = 2 \ \rightarrow t_1 = \ln 2 > 0 \end{cases} \qquad \text{(That is impossible for } t_1 > 0\text{)}$$

In this case;

$u^* = Ae^t = 2e^t$ Optimal control

$x_1^* = e^t$ Optimal path

Total cost is as follows:

$$J = \frac{1}{2}\int_0^{\ln 2}(u^*)^2 dt = \frac{1}{2}\int_0^{\ln 2} 4e^{2t} dt$$

$$= 2\int_0^{\ln 2} e^{2t} dt = 2(\frac{1}{2}e^{2t})_0^{\ln 2}$$

$$= 2(\frac{1}{2}e^{\ln 4} - \frac{1}{2}e^0) = 2(\frac{1}{2}(4) - \frac{1}{2})$$

$$= 2(2 - \frac{1}{2}) = 3$$

B) $a = 2, \ b = 1$

$$AB = 0 \begin{cases} B = 0 \Rightarrow e^{t_1} = \frac{1}{2} \Rightarrow t_1 = \ln\frac{1}{2} < 0 \\ or \\ A = 0 \Rightarrow e^{t_1} = 2 \end{cases}$$ (That is impossible for $t_1 > 0$)

In this case;

$u^* = 0$ Optimal control

$x_1^* = 2e^{-t_1}$ Optimal path

Total cost $J = 0$

Conclusion

This research guides us to Pontryagin's maximum principle that is used for solving problems with constraints or restraints existing on control or with state variables and also, dealing with restraints of inequality. Other applications of this principle are related to optimal form and description of a problem at two separated points with limited condition and when it is solved, explicit expression will be available for optimal control.

References

1. Kirk, Donald E,(1937). Optimal Control theory ; an introduction.

2. Differntial Equations with Applications and Historical Notes Georg F. Simmons by McGraw-Hill Inc., 1972

3. Louis Leithold,(1981). The Calculus with Analytic Geometry

4. Bartle, R.G., The Elements of Real Analysis. New York; John Wiley & Sons, Inc., 1964.

5. Gelfand, I.M., and S.V.Fomin, Calculus of Variation. Englewood Cliffs,N.J.; Prentice-Hall, Inc., 1963.